KCE Society’s COEIT, Jalgaon

Department of First year Engineering (Basic Science)

Engineering Mathematics-I

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Reference: Advanced Engineering Mathematics 10/e by ERWIN KREYSZIG (John Wiley and sons)

UNIT 1: Matrices
Eigen values and eigen vectors

A matrix eigenvalue problem considers the vector equation (1)

$$\mathbf{Ax} = \lambda \mathbf{x}.$$

Here $\mathbf{A}$ is a given square matrix, λ an unknown scalar, and $\mathbf{x}$ an unknown vector. In a matrix eigenvalue problem, the task is to determine λ's and $\mathbf{x}$'s that satisfy (1).

Since $\mathbf{x} = \mathbf{0}$ is always a solution for any and thus not interesting, we only admit solutions with $\mathbf{x} \neq \mathbf{0}$.

The solutions to (1) are given the following names: The λ's that satisfy (1) are called **eigenvalues of A** and the corresponding nonzero $\mathbf{x}$'s that also satisfy (1) are called **eigenvectors of A**.

- We formalize our observation. Let $\mathbf{A} = [a_{jk}]$ be a given nonzero square matrix of dimension $n \times n$. Consider the following vector equation:
- **(1)** $$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$
- The problem of finding nonzero $\mathbf{x}$'s and λ's that satisfy equation (1) is called an eigenvalue problem.

A value of λ for which (1) has a solution $\mathbf{x} \neq \mathbf{0}$ is called an **eigenvalue** or *characteristic value* of the matrix $\mathbf{A}$.

The corresponding solutions $\mathbf{x} \neq \mathbf{0}$ of (1) are called the **eigenvectors** or *characteristic vectors* of $\mathbf{A}$ corresponding to that eigenvalue λ.

The set of all the eigenvalues of $\mathbf{A}$ is called the **spectrum** of $\mathbf{A}$. We shall see that the spectrum consists of at least one eigenvalue and at most of n numerically different eigenvalues.

The largest of the absolute values of the eigenvalues of $\mathbf{A}$ is called the *spectral radius* of $\mathbf{A}$, a name to be motivated later.

How to Find Eigenvalues and Eigenvectors

EXAMPLE 1

Determination of Eigenvalues and Eigenvectors

We illustrate all the steps in terms of the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

Solution.
(a) ***Eigenvalues.*** These must be determined ***first***.
Equation (1) is

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

$$\begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2. \end{aligned}$$

Solution. ***(continued 1)***

(a) ***Eigenvalues.*** ***(continued 1)***

Transferring the terms on the right to the left, we get

$$\begin{aligned} (-5-\lambda)x_1 + \qquad\qquad 2x_2 &= 0 \\ 2x_1 + (-2-\lambda)x_2 &= 0 \end{aligned} \tag{2*}$$

This can be written in matrix notation

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0 \tag{3*}$$

Because (1) is $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, which gives (3*).

We see that this is a ***homogeneous*** linear system. By Cramer's theorem it has a nontrivial solution (an eigenvector of **A** we are looking for) if and only if its coefficient determinant is zero, that is,

$$D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0.$$

We call $D(\lambda)$ the **characteristic determinant** or, if expanded, the **characteristic polynomial**, and $D(\lambda) = 0$ the **characteristic equation** of $\mathbf{A}$. The solutions of this quadratic equation are $\lambda_1 = -1$ and $\lambda_2 = -6$. These are the eigenvalues of $\mathbf{A}$.

(b_1) ***Eigenvector of* A *corresponding to* $\boldsymbol{\lambda_1}$.** This vector is obtained from (2*) with $\lambda = \lambda_1 = -1$, that is,

$$
\begin{aligned}
-4x_1 + 2x_2 &= 0 \\
2x_1 - x_2 &= 0.
\end{aligned}
$$

Eigenvector of* A *corresponding to* λ_1. *(continued)

A solution is $x_2 = 2x_1$, as we see from either of the two equations, so that we need only one of them. This determines an eigenvector corresponding to $\lambda_1 = -1$ up to a scalar multiple. If we choose $x_1 = 1$, we obtain the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{Check:} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)x_1 = \lambda_1 x_1.$$

(b$_2$) ***Eigenvector of* A *corresponding to* λ_2.**
For $\lambda = \lambda_2 = -6$, equation (2*) becomes

$$x_1 + 2x_2 = 0$$
$$2x_1 + 4x_2 = 0.$$

A solution is $x_2 = -x_1/2$ with arbitrary x_1. If we choose $x_1 = 2$, we get $x_2 = -1$. Thus an eigenvector of **A** corresponding to $\lambda_2 = -6$ is

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{Check:} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = (-6)x_2 = \lambda_2 x_2.$$

This example illustrates the general case as follows. Equation (1) written in components is

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n .
\end{aligned}$$

Transferring the terms on the right side to the left side, we have

$$\begin{aligned}
(a_{11} - \lambda)x_1 + \quad a_{12}x_2 + \cdots + \quad a_{1n}x_n &= 0 \\
a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + \quad a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + \quad a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0.
\end{aligned}$$

In matrix notation,

(3)

By Cramer's theorem, this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

$\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix** and $D(\lambda)$ the **characteristic determinant** of $\mathbf{A}$. Equation (4) is called the **characteristic equation** of $\mathbf{A}$. By developing $D(\lambda)$ we obtain a polynomial of nth degree in λ. This is called the **characteristic polynomial** of $\mathbf{A}$.

Eigenvalues

The eigenvalues of a square matrix $\mathbf{A}$ *are the roots of the characteristic equation* (4) *of* $\mathbf{A}$.
Hence an $n \times n$ *matrix has at least one eigenvalue and at most* n *numerically different eigenvalues.*

The eigen**values** ***must be determined first***.
Once these are known, corresponding eigen*vectors* are obtained from the system (2), for instance, by the Gauss elimination, where λ is the eigenvalue for which an eigenvector is wanted.

Eigenvectors, Eigenspace

If $\mathbf{w}$ *and* $\mathbf{x}$ *are eigenvectors of a matrix* $\mathbf{A}$ *corresponding to* ***the same*** *eigenvalue* λ, *so are* $\mathbf{w} + \mathbf{x}$ (provided $\mathbf{x} \neq -\mathbf{w}$) *and* $k\mathbf{x}$ *for any* $k \neq 0$.

Hence the eigenvectors corresponding to one and the same eigenvalue λ *of* $\mathbf{A}$, *together with* $\mathbf{0}$, *form a vector space* (cf. Sec. 7.4), *called the* **eigenspace** *of* $\mathbf{A}$ *corresponding to that* λ.

In particular, *an eigenvector* $\mathbf{x}$ *is determined only up to a constant factor.*
Hence we can **normalize** $\mathbf{x}$, that is, multiply it by a scalar to get a unit vector

Ex. Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Solution.

For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

The roots (eigenvalues of **A**) are $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$.

To find eigenvectors, we apply the Gauss elimination (Sec. 7.3) to the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, first with $\lambda = 5$
and then with $\lambda = -3$. For $\lambda = 5$ the characteristic matrix is

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}.$$

It row-reduces to

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 2. Choosing $x_3 = -1$ we have $x_2 = 2$ from

$$-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$$

and then $x_1 = 1$ from $-7x_1 + 2x_2 - 3x_3 = 0$.

Hence an eigenvector of $\mathbf{A}$ corresponding to $\lambda = 5$ is $\mathbf{x}_1 = [1 \quad 2 \quad -1]^T$.

For $\lambda = -3$ the characteristic matrix

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

row-reduces to

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence it has rank 1.
From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing $x_2 = 1$, $x_3 = 0$ and $x_2 = 0$, $x_3 = 1$, we obtain two linearly independent eigenvectors of **A** corresponding to $\lambda = -3$ [as they must exist by (5), Sec. 7.5, with rank = 1 and n = 3],

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$